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Entropy of a Fibered Dynamical System

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Abstract

The entropy of the geodesic flow associated to a fibered dynamical system is shown to be zero; in particular the entropy of a quantizable dynamical system is zero. An ergodic dynamical system which defines a quantizable dynamical system is outlined.

1. *Introduction*

A dynamical system (DS) in the sense of Poincaré, Birkhoff and Reeb is a pair (M, C) where M is a connected differentiable manifold and C is a differentiable vector field on M which does not vanish everywhere. We will consider only *nonsingular* DSs, i.e., C is nonnull. A DS is *regular* if the foliation defined by the nonnull vector field C is regular in the sense of Palais (1957); and a DS is *proper* (or *complete*) if C generates a global 1-parameter group $\phi_t = \exp(tC)$ $(-\infty < t < \infty)$. Of course if M is compact, C is proper (Kobayashi & Nomizu, 1963, Prop. I.1.6). The *period function* of a DS is the function $\lambda(x) = \inf\{t > 0 | \phi_t(x) = x \text{ for } x \text{ in } M\}$. If λ is a finite constant, then the DS is called *finite*. A regular, proper finite DS is called a *fibered dynamical system* (FDS). The name arises from the following facts. If B is the orbit space, M/ϕ_t, for FDS (M, C), then $\xi: S^1 \to M \xrightarrow{p} B$ is a principal circle bundle by Palais (1957, Section I.5). Furthermore, for any principal circle bundle ξ there is a connection 1-form ω on M with curvature form $\Omega = d\omega$ and a unique integral 2-form Θ on B such that $p^*\Theta = \Omega$ (Hurt, 1973; Kobayashi, 1956). In addition the following result is easily shown:

Proposition (Tanno (1965)) 1. If (M, C) is a proper regular DS, then the following are equivalent:

(a) $\lambda(x)$ is a constant (finite or infinite);
(b) there exists a 1-form ω on M such that $\omega(C) = 1$ and $\mathscr{L}(C)\omega = 0$;
(c) there exists a Riemannian metric g on M such that $g(C, C) = 1$ and $\mathscr{L}(C)g = 0$.

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As we have noted in Hurt (1971a, Section 2) a contact manifold (M, ω) where ω is a 1-form satisfying $\omega \wedge (d\omega)^n \neq 0$ determines a DS (M, C) where C is the 'associated vector field' to ω. If this DS is regular proper and finite (M, C) is called a quantizable dynamical system (QDS) (Hurt, 1971a, 1971b, 1973). Clearly a QDS is a FDS. In addition, gauge invariant unified field structures (cf. Hurt, 1970, and 1973, Section 1) are FDSs.

By (c) in Proposition 1 we see that C is a unit Killing vector; thus each orbit of ϕ_t is a geodesic with respect to the metric g. And every Killing vector is *incompressible*—i.e., C leaves the volume element η of the Riemannian manifold (M, g) invariant (Sasaki, 1958). Thus the flow ϕ_t is a measure preserving transformation for the measure μ on M defined by η (Godbillon, 1969, Prop. 7.2.2)—i.e., $\mu_\eta(\phi_t A) = \mu_\eta(A)$ for every Borel set A on M and for every t in R. In other words,

Proposition 2. If (M, C) is a FDS, then (M, μ, ϕ_t) is a classical dynamical system in the sense of Arnold & Avez (1968, Section 1.1).

In the case (M, C) is a QDS, the volume form η is $\omega \wedge (d\omega)^n$ which is clearly invariant by C—i.e., $\mathscr{L}(C)\eta = 0$—since $\mathscr{L}(C)\omega = \mathscr{L}(C)\,d\omega = 0$ (cf. Hurt, 1971a, Godbillon, 1969, Prop. 7.5.7).

2. *Entropy*

We review briefly the notations of entropy from Arnold & Avez (1968). Let $z(t)$ denote the function on $[0,1]$ defined by

$$z(t) = \begin{cases} -t\log t & \text{if } 0 < t \leqslant 1 \\ 0 & \text{if } t = 0 \end{cases}$$

Let α be a finite (so measurable) partition of M—i.e., a finite collection of nonempty nonintersecting measurable subsets $\{A_i\}_{i\in I}$ of M for which $\mu(A_i \cap A_j) = 0$ if $i \neq j$ and $\mu(M - \bigcup_{i\in I} A_i) = 0$. Let F denote the set of finite partitions of M. The *sum* of two partitions α, β in F is $\alpha \vee \beta = \{A_i \cap A_j \mid A_i$ in α, β_j in $\beta\}$. We say β is a *refinement* of α denoted $\alpha \leqslant \beta$, if for every B_j in β there exists an A_i in α such that $\mu(B_j - B_j \cap A_i) = 0$. The *entropy of a partition* $\alpha = \{A_i\}_{i\in I}$ in F is $h(\alpha) = \sum_{i\in I} z(\mu(A_i))$.

Proposition (Arnold & Avez, 1968, 12.12) 2.1. If $\alpha \leqslant \beta$ then $h(\alpha) \leqslant h(\beta)$.

If ϕ is an automorphism of measure space (M, μ) (for definition see Arnold & Avez, 1968, App. 6), then $\phi\alpha = \{\phi(A_i)\}_{i\in I}$. Then since ϕ is measure preserving,

Proposition 2.2. $h(\phi\alpha) = h(\alpha)$.

The *entropy of a partition* α with respect to an automorphism ϕ is

$$h(\alpha, \phi) = \lim_{n\to\infty} \frac{h(\alpha \vee \phi\alpha \vee \ldots \vee \phi^{n-1}\alpha)}{n}, \qquad n \in \mathbf{Z}^+$$

The *entropy of an automorphism* ϕ is then

$$h(\phi) = \sup_{\alpha \in F} h(\alpha, \phi)$$

Clearly $h(\phi) \geqslant 0$. Furthermore

Proposition (Arnold & Avez, 1968, 12.24) 2.3. $h(\phi)$ is an invariant of the abstract dynamical system (M, μ, ϕ).

If (M, μ, ϕ_t) is a classical dynamical system, then ϕ_t is an automorphism of (M, μ) for each fixed t; so for each fixed $t, h(\phi_t)$ is defined; and $h(\phi_t)$ satisfies:

Proposition (Abramov, 1959) 2.4. $h(\phi_t) = |t|\, h(\phi_1)$ for all t in R.

Thus the natural definition for *entropy* of a flow of classical DS (M, μ, ϕ_t) is $h(\phi_1)$.

Let (M, μ, ϕ_t) be the classical DS defined by FDS (M, C). Clearly the period λ can be chosen to be an integer (by modifying C). Then $\phi_1^\lambda(x) = \phi_\lambda(x) = x$. Thus for suitably large n,

$$\alpha \vee \phi_1 \alpha \vee \ldots \vee \phi_1^{n-1} \alpha \leqslant \alpha \vee \phi_1 \alpha \vee \ldots \vee \phi_1^{\lambda-1} \alpha$$

So by Proposition 2.1 and Proposition 2.2,

$$h(\alpha, \phi_1) = \lim_{n\to\infty} \frac{h(\alpha \vee \phi_1 \alpha \vee \ldots \vee \phi_1^{n-1} \alpha)}{n} \leqslant \lim_{n\to\infty} \frac{\lambda h(\alpha)}{n} = 0$$

and

$$h(\phi_1) = \sup_{\alpha \in F} h(\phi_1, \alpha) = 0$$

By Abramov's Theorem (Prop. 2.4) we have:

Proposition 2.5. If (M, C) is a FDS with geodesic flow ϕ_t, then $h(\phi_t) = 0$ for all t in R. In particular the entropy of a QDS is zero.

3. *An Example*

Let (M, μ, ϕ_t) be a classical DS in the sense of Arnold & Avez (1968) where M is a three-dimensional manifold and ϕ_t is an ergodic flow. Assume the vector field associated to ϕ_t is finite in the sense of Section 1; and assume there are two C^∞ differentiable eigenfunctions f_1, f_2 of ϕ_t whose eigenvalues are linearly independent over the integers Z. Then it follows directly (Niwa, 1969) that $\{df_1, df_2\}$ are linearly independent everywhere. Let $\theta_k = (1/2\pi i)\log f_k \in T^1$. Then $p: M \to T^2 : p(x) = (\theta_1, \theta_2)$ is of maximal rank. This fibration is easily shown to be locally trivial (Niwa, 1969). And T^2 is a Hodge manifold. Thus we have by Niwa (1969).

Proposition 3.1. If (M, μ, ϕ_t) is an ergodic classical DS as above, then $\xi: S^1 \to M \to B = T^2$ is a principal circle bundle over a Hodge manifold (so a QDS). Furthermore the flow induced by ϕ_t on B is the Jacobi (or quasi-periodic) flow (Arnold & Avez, 1968, App. 1).

A more precise characterization of these dynamical systems would be helpful, in particular a generalization to circle bundles (or QDSs) over abelian varieties; also a further study of induced flows in principal bundles is needed.

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